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Generalized Second Law of Thermodynamics in $f(T)$ Cosmology with Power-Law and Logarithmic Corrected Entropies

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Abstract: We study the generalized second law (GSL) of thermodynamics in $f(T)$ cosmology. We consider the universe as a closed bounded system filled with n component fluids in the thermal equilibrium with the cosmological boundary. We use two different cosmic horizons: the future event horizon and the apparent horizon. We show the conditions under which the GSL will be valid in specific scenarios of the quintessence and the phantom energy dominated eras. Further we associate two different entropies with the cosmological horizons: with a logarithmic correction term and a power-law correction term. We also find the conditions for the GSL to be satisfied or violated by imposing constraints on model parameters.

I.   INTRODUCTION

A unification of quantum mechanics (QM) and general relativity (GR) is one of the goals of modern physics. Attempts to unify these two theories are termed Quantum Gravity in the literature. However a consistent theory of quantum gravity is not discovered yet, although several theories like loop quantum gravity and string theory predict new features about black holes and singularities. A well-known result is that general relativity predicts gravitational collapse of stars to black holes while quantum mechanics (together with GR) predicts their evaporation. Both string theory and loop quantum gravity predict that a black hole emits thermal radiations whose thermal spectrum might deviate from the Planck black body spectrum at a certain small scale [1], with a temperature proportional to its surface gravity at the black hole horizon and with an entropy proportional to its horizon area, the same predictions made by Hawking and Bekenstein several decades ago [2]. The Hawking temperature and horizon entropy together with the black hole mass obey the first law of thermodynamics. Some important discoveries connecting laws of thermodynamics and the Einstein field equations have been made [3]. It has also been demonstrated that by applying the first law of thermodynamics to the apparent horizon of a Friedmann-Lemaître-Robertson-Walker (FLRW) universe and assuming the geometric entropy given by a quarter of the apparent horizon area, the Friedmann equations of the universe

with any spatial curvature can be derived [4]. Similar results also hold in scalar tensor, Gauss-Bonnet, Lovelock and $f(R)$ gravities [5].

In this paper, we explore the GSL in the cosmological context and it can be interpreted that the time derivative generalized entropy (which has to be the sum of entropy of all fluids filling the space along with the entropy of the cosmological horizon) must be increasing (or non-decreasing) function of time. There have been some attempts to prove the GSL in different ways: a simple direct explicit proof of the GSL of black hole thermodynamics was proved for a quasistationary semiclassical black hole in Ref. [6], also for two dimensional black hole spacetime [7], using charged and rotating black holes [8], by quantum information theory [9], using adiabatically collapsing thick light shells [10]. However the GSL may be violated in certain cases: It has been shown that classical non-minimally coupled scalar fields can violate all of the standard energy conditions and the GSL [11]. The GSL has found some interesting implications in string cosmology as well where it forbids singular string cosmologies [12]. In the study of accretion of phantom energy on black holes in cosmological background, the black holes will completely vanish if the GSL is violated [13]. In the framework of Horava-Lifshitz cosmology, it has been shown that under detailed balance the GSL is generally valid for flat and closed geometry and it is conditionally valid for an open universe, while beyond detailed balance the GSL is only conditionally valid for all curvatures [14]. In $F(R, G)$ gravity and modified gravity, the conditions for validity of the GSL have been discussed in [15] while in chameleon cosmology, the validity of the GSL in flat FRW chameleon cosmology where the boundary of the universe is the dynamical apparent horizon [16] has been investigated. The GSL for fractional action cosmology by choosing various forms of scale factors was studied and

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proved that the GSL is not valid in all scenarios. For cosmological event horizon, it has been proved that the GSL is satisfied [18]. The conditions of validity of generalized second law in phantom energy dominated era has also been studied [19]. However the GSL is violated when a black hole is introduced in a phantom energy dominated universe [20]. On the other hand, the GSL can be saved from the violation provided the temperature is not taken as de Sitter temperature [21]. In a quintom (i.e. quintessence and phantom) dominated universe, the conditions for the GSL validation were investigated [22]. For a holographic dark energy dominated universe, the GSL is respected for certain values of deceleration parameter [23]. Moreover, the GSL has been studied in some lower-dimensional cosmological settings with several interesting consequences [24]. The validity of GSL of thermodynamics has been investigated in the cosmological scenario where dark energy interacts with both dark matter and radiation [25]. It has been shown that the GSL is always and generally valid, independently of the specific interaction form of the fluids equation of state and of the background geometry.

The power-law correction to entropy which appears in dealing with the entanglement of quantum fields in and out the horizon is given by [26]

$$S_A = \frac{A}{4}\Big(1 - K_\alpha A^{1-\frac{\alpha}{2}}\Big), \qquad (1)$$

where α is a dimensionless constant and a free parameter, and

$$A = 4\pi R_h^2, \quad K_\alpha = \frac{\alpha(4\pi)^{\frac{\alpha}{2}-1}}{(4-\alpha)r_c^{2-\alpha}}, \qquad (2)$$

where r_c is a cross-over scale, R_h is the radius, and A is the area of the cosmological horizon. For entropy to be a well-defined quantity, we require $\alpha > 0$. The second term in (1) can be regarded as a power-law correction to the area law, resulting from entanglement, when the wavefunction of the field is chosen to be a superposition of ground state and exited state [27]. Several aspects of power-law corrected entropy (1) have been studied in the literature including the GSL [28], power-law entropy corrected models of dark energy [29].

The quantum corrections provided to the entropy-area relationship leads to the curvature correction in the Einstein-Hilbert action and vice versa. The logarithmic corrected entropy is [30]

$$S_A = \frac{A}{4} + \beta \log\Big(\frac{A}{4}\Big) + \gamma. \qquad (3)$$

These corrections arise in the black hole entropy in loop quantum gravity due to thermal equilibrium fluctuations and quantum fluctuations [31]. It has been shown [32] that in a (super) accelerated universe the GSL is valid whenever $\beta(<) > 0$ leading to a (negative) positive contribution from logarithmic correction to the entropy. In case of super acceleration the temperature of the dark

energy is obtained to be less or equal to the Hawking temperature. Using the corrected entropy-area relation motivated by the loop quantum gravity, the validity of the GSL in the FRW universe filled with an interacting viscous dark energy with dark matter and radiation was examined. It has been found that the GSL is always satisfied throughout the history of the universe for any spatial curvature regardless of the dark energy model.

However in $f(T)$ gravity, the entropy-area relation gets a modification like $A \to A(1 + f_T)$, assuming $f_{TT} << 1$ [34]. Here $f_T \equiv \frac{df}{dT}$, $f_{TT} \equiv \frac{d^2f}{dT^2}$, and T is the torsion scalar. The logarithmic corrected entropy becomes

$$S_A = \frac{A(1 + f_T)}{4} + \beta \log\Big(\frac{A(1 + f_T)}{4}\Big) + \gamma. \qquad (4)$$

For the entropy to be a well-defined quantity, we must have $f_T > 0$. Similarly the power-law corrected entropy becomes

$$S_A = \frac{A(1 + f_T)}{4}\Big(1 - K_\alpha(A(1 + f_T))^{1-\frac{\alpha}{2}}\Big). \qquad (5)$$

Incidentally, thermodynamics with the apparent horizon and the Wald entropy [36] in $f(R)$ gravity [37, 38], various modified gravitational theories [39, 40], and $f(T)$ gravity [41] has been explored.

The plan of the paper is as follows: In section II, we write down the Friedmann equations and the laws of thermodynamics for our further use. In section III, we study the GSL using the classical Bekenstein-Hawking entropy-area relation with the apparent and event horizons. In sections IV and V, we perform similar analysis for the power-law and logarithmic corrected entropy-area relations. In section VI, we digress to non-equilibrium situation and finally discuss our results briefly in section VII. In all sections, we choose units $c = 1 = G$.

II. BASIC EQUATIONS

A. $f(T)$ gravity

If we accept the equivalence principle, we must work with a curved manifold for the construction of a gauge theory for gravitational field. It is not necessary to use the Riemannian manifolds. The general form of a gauge theory for gravity, with metric, non-metricity and torsion can be constructed easily [35]. If we relax the non-metricity, our theory is defined on Weitzenböck spacetime, with torsion and with zero local Riemann tensor. In this theory, which is called teleparallel gravity, we use a non-Riemannian spacetime manifold. The dynamics of the metric determined using the torsion T. The basic quantities in teleparallel or the natural extension of it, namely $f(T)$ gravity is the vierbein (tetrad) basis e^i_μ [42, 43]. This basis is an orthonormal, coordinate free basis, defined by the following equation

$$g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij}.$$

This tetrad basis must be orthonormal and η_{ij} is the Minkowski flat tensor. It means that $e^i_\mu e_{\mu(j)} = \delta^i_j$. One suitable form of the action for $f(T)$ gravity in Weitzenböck spacetime is given by [44]

$$S = \int d^4x e\Big(\frac{1}{16\pi}F(T) + L_m\Big), \qquad (6)$$

where $F(T) = T + f(T)$ is the Lagrangian for torsion gravity and f is an arbitrary function, $e = \det(e^i_\mu)$. The dynamical quantity of the model is the scalar torsion T and the matter Lagrangian L_m. Here T is defined by

$$T = S_\rho{}^{\mu\nu}T^\rho{}_{\mu\nu},$$

with

$$T^\rho{}_{\mu\nu} = e^\rho_i(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu),$$

$$S_\rho{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\theta\nu}{}_\theta - \delta^\nu_\rho T^{\theta\mu}{}_\theta),$$

where the asymmetric tensor (which is also called the contorsion tensor) $K^{\mu\nu}{}_\rho$ reads

$$K^{\mu\nu}{}_\rho = -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}).$$

The equation of motion derived from the action, by varying with respect to the e^i_μ, is given by

$$e^{-1}\partial_\mu(eS_i{}^{\mu\nu})(1 + f_T) - e^\lambda_i T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}f_T$$

$$+S_i{}^{\mu\nu}\partial_\mu(T)f_{TT} - \frac{1}{4}e^\nu_i(1 + f(T)) = 4\pi G e^\rho_i T_\rho{}^\nu,$$

where $T_{\mu\nu}$ is the energy-momentum tensor for matter sector of the Lagrangian L_m, defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\Big(\int d^4x\sqrt{-g}L_m\Big)}{\delta g^{\mu\nu}}.$$

It is a straightforward calculation to show that this equation of motion is reduced to the Einstein gravity when $f(T) = 0$. Indeed, this is the equivalence between the teleparallel theory and the Einstein gravity [45].

We take a spatially flat homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (7)$$

where $a(t)$ is the scale factor and $d\theta^2 + \sin^2\theta d\phi^2 \equiv d\Omega^2$ is the metric of two-dimensional sphere with unit radius. The Friedmann equations in $f(T)$ gravity are

$$H^2 = \frac{8\pi}{3}\rho - \frac{1}{6}f - 2H^2 f_T, \qquad (8)$$

$$\dot{H} = -\frac{4\pi(\rho + p)}{1 + f_T - 12H^2 f_{TT}}. \qquad (9)$$

Here $H = \dot{a}/a$ is the Hubble parameter and the dot denotes the time derivative of $\partial/\partial t$. In addition, $\rho = \sum_i \rho_i$ and $p = \sum_i p_i$ $(i = 1 \ldots n)$ are the total energy density and pressure of n cosmic fluids, respectively. For the FLRW metric (7), the trace of the torsion tensor is $T = -6H^2$, and hence Eqs. (8) and (9) are simplified as

$$H^2 = \frac{1}{1 + 2f_T}\Big(\frac{8\pi}{3}\rho - \frac{f}{6}\Big), \qquad (10)$$

$$\dot{H} = -4\pi\Big(\frac{\rho + p}{1 + f_T + 2Tf_{TT}}\Big). \qquad (11)$$

We will work under the assumption $f_{TT} << 1$. Thus Eq. (11) is reduced to

$$\dot{H} \simeq -4\pi\Big(\frac{\rho + p}{1 + f_T}\Big). \qquad (12)$$

Moreover, the energy conservation equation is given by

$$\dot{\rho} + 3H(\rho + p) = 0. \qquad (13)$$

B. Generalized second law (GSL)

The GSL of thermodynamics for black holes states that entropy of a black hole added with the entropy of the background universe must be non-decreasing. In other words, the generalized entropy function must be positive definite: $S \equiv \sum_i S_i + S_h \geq 0$, where S_i and S_h are the entropies of individual component and the event horizon. It is proved in Ref. [34] that the usual first law of thermodynamics does not hold in $f(T)$ gravity and an extra term due to 'entropy production' S_p is introduced to the first law. Therefore to study the GSL, we use the 'modified first law of thermodynamics' [34]

$$T_i dS_i = d(\rho_i V) + p_i dV - T_i dS_p, \qquad (14)$$

where T_i is the temperature and S_i is the entropy of the ith component of the fluid, and also $V = \frac{4\pi}{3}R_h^3$ is the volume of the horizon.

We assume the n-component fluid to be interacting and exchange energy. Although this assumption is purely phenomenological, it helps in resolving certain problems in cosmology including a coincidence problem [46] and is also consistent with the astrophysical observations [47]. The continuity equation (13) for each fluid reads

$$\dot{\rho}_i + 3H(\rho_i + p_i) = Q_i, \qquad (15)$$

where Q_i are the interaction terms, collectively satisfying $\sum_i Q_i = 0$. From Eqs. (14) and (15), we get

$$\dot{S}_i = \frac{4\pi}{3}R_h^3\frac{Q_i}{T_i} + 4\pi R_h^2(\dot{R}_h - HR_h)(\frac{\rho_i + p_i}{T_i}) - \dot{S}_p. \quad (16)$$

We suppose the thermal equilibrium, i.e., $T_i = T_H$, and thus

$$\dot{S}_I + \dot{S}_p = -\frac{R_h^2}{T_H}(\dot{R}_h - HR_h)\dot{H}(1 + f_T), \qquad (17)$$

where $S_I = \sum_i S_i$ is the total internal entropy of $n-$ fluids. The form of GSL in this work reads

$$\dot{S}_{tot} = \dot{S}_I + \dot{S}_p + \dot{S}_h \geq 0\,.$$

Here we introduce two definitions of cosmological horizons from the literature: the *dynamical apparent horizon* is a null surface with vanishing expansion. In the flat FLRW universe, it is $R_A = H^{-1}$ (which is also called Hubble horizon). The second interesting case is the *future event horizon* which is the distance that light travels from the present time to infinity and is defined as

$$R_E = a(t) \int\limits_t^\infty \frac{dt'}{a(t')}\,, \quad \dot{R}_E = HR_E - 1\,. \qquad (18)$$

III. GSL WITH BEKENSTEIN-HAWKING ENTROPY-AREA RELATION

We consider the form of entropy in $f(T)$ gravity

$$S_A = S_A(X)\,, \quad X \equiv \frac{A(1 + f_T)}{4}\,. \qquad (19)$$

Differentiating (19) with respect to (w.r.t.) t, we get

$$\dot{S}_A = \left[\frac{(1 + f_T)\dot{A}}{4} + \frac{A}{4} f_{TT}\dot{T} \right] \frac{dS}{dX}\,. \qquad (20)$$

Ignoring f_{TT}, we obtain

$$\dot{S}_A \simeq \frac{(1 + f_T)\dot{A}}{4} \frac{dS}{dX}\Big|_X\,. \qquad (21)$$

Bekenstein-Hawking entropy in $f(T)$ gravity is described by

$$S_A(X) = X\,. \qquad (22)$$

Hence (21) implies

$$\dot{S}_A = \frac{(1 + f_T)\dot{A}}{4}\,. \qquad (23)$$

A. Use of the apparent horizon

It has been shown [48] that in an accelerating universe, the GSL holds only in the case that the boundary surface is the apparent horizon, not in the case of the event horizon. This suggests that event horizon is not a physical boundary from the point of view of thermodynamics.

We examine the dynamical apparent horizon [4]

$$R_A = \frac{1}{H}\,. \qquad (24)$$

The apparent horizon R_A is a marginally trapped surface with vanishing expansion and is determined from the condition $g^{ij}\partial_i \tilde{r} \partial_j \tilde{r} = 0$, where $\tilde{r} = r(t)a(t)$ and $i, j = 0, 1$ [49]. Assuming $A = 4\pi R_A^2$, we have

$$\dot{S}_A = -2\pi \frac{\dot{H}}{H^3}(1 + f_T)\,. \qquad (25)$$

Clearly by substituting $f_T = 0$, we obtain results for the usual Hawking entropy. The form of the GSL expression here becomes

$$\dot{S}_{tot} \equiv \dot{S}_A + \dot{S}_I + \dot{S}_p = 2\pi \frac{\dot{H}^2}{H^5}(1 + f_T) \geq 0\,, \qquad (26)$$

where we have assumed the thermal equilibrium for our dynamical system $T_i = T_H$ and used $T_H = \frac{H}{2\pi}$ [50].

We notice that the assumption of the thermal equilibrium in cosmological setting is very ideal, because major components of the universe including dark matter, dark energy and radiation (cosmic microwave background (CMB) and neutrinos) have entirely different temperatures [51]. However it has been found that the contribution of the heat flow between dark energy and dark matter for the GSL in the nonequilibrium thermodynamics is very small as $O(10^{-7})$ [52]. Therefore the equilibrium thermodynamics is still preserved. Further, if there is any thermal difference between the fluids and the horizon, the transfer of energy across the horizon might change the geometry of the horizon and the FLRW spacetime [32]. We mention that thermodynamics with the apparent horizon and the Wald entropy in a model of $f(R)$ gravity realizing the crossing of the phantom divide [53], which can occur also in $f(T)$ gravity [54], has been studied [37].

B. Use of the event horizon

Time derivative of the entropy of the future event horizon ($S_A = \pi R_E^2(1 + f_T)$) is given by

$$\dot{S}_A = 2\pi R_E \dot{R}_E(1 + f_T)\,. \qquad (27)$$

The total entropy for the GSL becomes

$$\dot{S}_{tot} = 2\pi R_E \dot{R}_E(1 + f_T) + \frac{2\pi R_E^2}{bH}\dot{H}(1 + f_T) \geq 0\,, \quad (28)$$

where we have provided the thermal equilibrium for our dynamical system and used $T_H = \frac{bH}{2\pi}$ with b being a constant [50]. Using the relation $\dot{R}_E = HR_E - 1$, we rewrite (28) in the following form

$$\dot{S}_{tot} \equiv 2\pi R_E^2(1 + f_T)\frac{d}{dt}\left[\log(R_E H^{\frac{1}{b}}) \right] \geq 0\,. \qquad (29)$$

Since $f_T > 0$, there is only needed to check the positivity of the logarithmic term. We choose the following set of the parameters [55]:

$$a(t) = a_0(t_s - t)^{-n}, \quad H = \frac{n}{t_s - t}, \quad R_E = \frac{t_s - t}{n + 1}, \quad (30)$$

where $a_0 > 0$, $n > 0$ and $t_s > t > 0$ with t_s being the time of occurrence of a future cosmic singularity like big rip. By the direct substitution of these equations into (29), we find that the following range of b must be valid

$$b \leq 1. \tag{31}$$

IV. GSL OF THERMODYNAMICS WITH POWER-LAW ENTROPY CORRECTION

The form of entropy with the power-law correction term is given by

$$S(X) = X[1 - K_\alpha(4X)^{1-\frac{\alpha}{2}}], \tag{32}$$

whose time derivative becomes

$$\dot{S} \simeq \frac{\dot{A}(1+f_T)}{4}\Big[1 - K_\alpha\Big(A(1+f_T)\Big)^{1-\frac{\alpha}{2}} \\ - K_\alpha\Big(1-\frac{\alpha}{2}\Big)\Big(A(1+f_T)\Big)^{1-\frac{\alpha}{2}}\Big]. \tag{33}$$

A. Using the apparent horizon

Taking time derivative of the entropy with the power-law correction and using the apparent horizon, we get

$$\dot{S}_A = -2\pi\frac{\dot{H}}{H^3}(1+f_T)\Big[1 - \\ K_\alpha\Big(2-\frac{\alpha}{2}\Big)\Big(\frac{4\pi}{H^2}(1+f_T)\Big)^{1-\frac{\alpha}{2}}\Big]. \tag{34}$$

Moreover the total entropy for the GSL is obtained by adding (34) to (16), we acquire

$$\dot{S}_{tot} = 2\pi\frac{\dot{H}}{H^3}(1+f_T)\Big[\frac{\dot{H}}{H^2} \\ + K_\alpha\Big(2-\frac{\alpha}{2}\Big)\Big(\frac{4\pi}{H^2}(1+f_T)\Big)^{1-\frac{\alpha}{2}}\Big] \geq 0. \tag{35}$$

We rewrite (35) in the following form

$$\dot{S}_{tot} = 2\pi\frac{\dot{H}}{H^3}(1+f_T)\Big[\frac{\dot{H}}{H^2} \\ + \frac{\alpha}{2}(Hr_c)^{\alpha-2}(1+f_T)^{1-\frac{\alpha}{2}}\Big] \geq 0. \tag{36}$$

There exist two special cases: In Quintessence [the non-Phantom] ($\dot{H} < 0$) phase,

- for $\alpha > 0$, we find $\dot{H} \leq -\frac{\alpha H^2}{2}(Hr_c)^{\alpha-2}(1+f_T)^{1-\frac{\alpha}{2}}$.

- for $\alpha < 0$, we obtain $\dot{S}_{tot} \geq 0$ for any $H, \dot{H}, \alpha$.

On the other hand, in the Phantom phase ($\dot{H} > 0$),

- for $\alpha < 0$, we acquire $\dot{H} \geq -\frac{\alpha H^2}{2}(Hr_c)^{\alpha-2}(1+f_T)^{1-\frac{\alpha}{2}}$.

- for $\alpha > 0$, we have $\dot{S}_{tot} \geq 0$ for any $H, \dot{H}, \alpha$.

B. Using the event horizon

For the power-law entropy, the time derivative of the horizon entropy is written as

$$\dot{S}_E = 2\pi R_E\dot{R}_E(1+f_T)\Big[1 \\ - K_\alpha(2-\frac{\alpha}{2})\Big(4\pi R_E^2(1+f_T)\Big)^{1-\frac{\alpha}{2}}\Big], \tag{37}$$

and the form of the GSL becomes

$$\dot{S}_{tot} = 2\pi R_E\dot{R}_E(1+f_T)\Big[1 - K_\alpha\Big(2-\frac{\alpha}{2}\Big) \\ \times \Big(4\pi R_E^2(1+f_T)\Big)^{1-\frac{\alpha}{2}}\Big] + \frac{2\pi R_E^2}{bH}\dot{H}(1+f_T) \geq 0. \tag{38}$$

We rewrite (38) in the following form

$$\dot{S}_{tot} = 2\pi R_E(1+f_T)\Big[\dot{R}_E(1-\frac{\alpha}{2}(R_Er_c)^{2-\alpha}) \\ \times (1+f_T)^{1-\frac{\alpha}{2}} + \frac{\dot{H}}{bH}R_E\Big] \geq 0. \tag{39}$$

- For the Quintessence phase: $\dot{H} < 0$, $\dot{R}_E > 0$, we have

$$\frac{\dot{R}_E}{R_E} \geq -\frac{\dot{H}}{bH}\Big(1-\frac{\alpha}{2}(R_Er_c)^{2-\alpha}(1+f_T)^{1-\frac{\alpha}{2}}\Big). \tag{40}$$

- For the Phantom phase: $\dot{H} > 0$, $\dot{R}_E < 0$, we acquire

$$\frac{\dot{R}_E}{R_E} \leq -\frac{\dot{H}}{bH}\Big(1-\frac{\alpha}{2}(R_Er_c)^{2-\alpha}(1+f_T)^{1-\frac{\alpha}{2}}\Big). \tag{41}$$

V. GSL OF THERMODYNAMICS WITH LOGARITHMIC CORRECTION

The entropy with the logarithmic correction is expressed as

$$S(X) = X + \beta\log(X) + \gamma, \tag{42}$$

where β and γ are constants. Upon the differentiation w.r.t t, we get

$$\dot{S} \simeq \frac{\dot{A}(1+f_T)}{4}\Big(1 + \frac{4\beta H^2}{A(1+f_T)}\Big). \tag{43}$$

A. Case of the apparent horizon

For the apparent horizon, (43) reduces to

$$\dot{S}_A = \frac{-2\pi\dot{H}}{H^3}(1+f_T)\Big(1 + \frac{4\beta H^2}{\pi(1+f_T)}\Big), \tag{44}$$

while the total entropy for the GSL reads

$$\dot{S}_{tot} = \frac{-2\pi\dot{H}}{H^3}(1+f_T)\Big(\frac{\beta H^2}{\pi(1+f_T)} - \frac{\dot{H}}{H^2}\Big) \geq 0. \quad (45)$$

Let us concentrate on only two special cases of interest: In the Quintessence phase ($\dot{H} < 0$),

- for $\beta > 0$, we find $\dot{S}_{tot} \geq 0$ in any time.

- for $\beta < 0$, we obtain $\dot{H} \leq \frac{\beta H^4}{\pi(1+f_T)}$ in any time.

In the Phantom phase ($\dot{H} > 0$):

- for $\beta < 0$, we have $\dot{S}_{tot} \geq 0$ in any time.

- for $\beta > 0$, we acquire $\dot{H} \geq \frac{\beta H^4}{\pi(1+f_T)}$ in any time.

B. Case of the event horizon

The definition of entropy for the comic event horizon with the logarithmic correction converts (43) to

$$\dot{S}_A = 2\pi R_E \dot{R}_E (1+f_T)\Big(1 + \frac{\beta}{\pi R_E^2(1+f_T)}\Big), \quad (46)$$

and in this case the corresponding form of the GSL becomes

$$\dot{S}_{tot} = 2\pi R_E \dot{R}_E (1+f_T)\Big(1 + \frac{\beta}{\pi R_E^2(1+f_T)}\Big)$$
$$+ \frac{2\pi R_E^2}{bH}\dot{H}(1+f_T) \geq 0. \quad (47)$$

- In the Quintessence era, $\dot{H} < 0$, $\dot{R}_E > 0$, we find

$$\frac{\dot{R}_E}{R_E} \geq -\frac{\dot{H}}{bH}\Big(1 + \frac{\beta}{\pi R_E^2(1+f_T)}\Big)^{-1}. \quad (48)$$

- In the Phantom era, $\dot{H} > 0$, $\dot{R}_E < 0$, we obtain

$$\frac{\dot{R}_E}{R_E} \leq -\frac{\dot{H}}{bH}\Big(1 + \frac{\beta}{\pi R_E^2(1+f_T)}\Big)^{-1}. \quad (49)$$

VI. DISCUSSION

In the present paper, we have investigated the validity of the GSL in $f(T)$ gravity. We have used the power-law

and logarithmic corrected forms of entropy for the cosmological horizon. An important point we have made is that the classical entropy-area law gets modification after adding the term $(1+f_T)$ to its expression. Since $f_T > 0$, it implies that the usual entropy of the horizon increases by the corresponding factor $(1+f_T)$, although the result is analogous to that in $f(R)$ gravity. However the crucial difference between thermodynamics in $f(R)$ and $f(T)$ gravities is that the first law of thermodynamics does not hold in $f(T)$ unlike $f(R)$. Nevertheless, after adding an entropy production term, the modified form of the first law becomes valid for thermodynamic studies. We have performed our analysis for the apparent and event horizons separately. In the usual Bekenstein-Hawking area law in $f(T)$ cosmology, for both of the Quintessence and Phantom regimes, we have found general conditions for the validity of the GSL. In the case of the event horizon, we have explicitly demonstrated that if the constant of the proportionality is $b \leq 1$, then the GSL remains valid. For the power law corrected form of the entropy, again we have shown that for either the Phantom or Quintessence dominated eras, the GSL is valid for any value of α. In the case of the event horizon, we have found that the validity of the GSL depends on the rate of the change $\frac{d\log(R_E)}{dt}$. For the logarithmic corrected form of the entropy, we have found that with the apparent horizon in both of the Phantom and Quintessence regimes, the GSL is valid. In the case of the event horizon, same as the power law corrected case, the validity of the GSL depends on the time derivative of the $\log(R_E)$. Our results are generally valid regardless the form of $f(T)$. As a result, our work has clarified some thermodynamic features of the $f(T)$ gravity.

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